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Trace formula and Spectral Riemann Surfaces for a class of tri-diagonal matrices

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Abstract

For tri-diagonal matrices arising in the simplified Jaynes–Cummings model, we give an asymptotics of the eigenvalues, prove a trace formula and show that the Spectral Riemann Surface is irreducible.

1. Introduction

We consider one-sided tri-diagonal matrices of the form $L + zB$, where

$$L = \begin{bmatrix} q_1 & 0 & 0 & 0 & \cdot \\ 0 & q_2 & 0 & 0 & \cdot \\ 0 & 0 & q_3 & 0 & \cdot \\ 0 & 0 & 0 & q_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \quad B = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdot \\ c_1 & 0 & b_2 & 0 & \cdot \\ 0 & c_2 & 0 & b_3 & \cdot \\ 0 & 0 & c_3 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.1}$$

and study their spectra in the case where the diagonal matrix majorizes the off-diagonal one in the sense of the following condition (or some version of it):

$$|q_k| \to \infty, \qquad \frac{(|b_k| + |c_k|)^2}{|q_k q_{k+1}|} \to 0. \tag{1.2}$$

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There is a vast literature (see [6,13,14,29] and the bibliography therein) devoted to a broad range of questions on these matrices and the corresponding operators in $\ell^2(\mathbb{N})$. We will be concerned with the following three questions:

1. Spectra $Sp(L + zB)$. Of course, $Sp(L) = \{q_k,\ k = 1, 2, \ldots\}$ and

$$Le_k = q_k e_k, \quad k = 1, 2, \ldots,$$

where $\{e_k\}_1^\infty$ is the canonical orthonormal basis in $\ell^2(\mathbb{N})$. Under condition (1.2), the spectrum $Sp(L + zB)$ is discrete as well (see, e.g., [5, Lemma 8] or [12,32]), and

$$Sp(L + zB) = \{E_n(z)\}_1^\infty,$$

where, for each n, $E_n(z)$ is an analytic function at least for small $|z|$, i.e., in the disk $|z| < R_n$ for some $R_n > 0$.

(1.A) *How large could R_n be chosen*?

Let us mention that in the case of Mathieu operator Volkmer [37] proved that $R_n \asymp n^2$ (see further discussion in Sections 7.1–7.3).

(1.B) *What is the asymptotic behavior of $E_n(z)$ if z is bounded, say $|z| \leqslant R$, and $n \to \infty$*?

2. Under conditions (1.2) and some further assumptions on the sequences q, b and c, one can introduce the regularized trace

$$tr(z) = \sum_{n=1}^{\infty} (E_n(z) - q_n)$$

as an entire function—see Definition in Section 5.4.

Can we evaluate it in specific examples?

3. Spectral Riemann Surface (SRS) of the pair $(L, B) \in$ (1.1), (1.2) is defined as

$$G = \{(\lambda, z) \in \mathbb{C}^2 : (L + zB)f = \lambda f,\ f \in \ell^2(\mathbb{N}),\ f \neq 0\}.$$

F.W. Schäfke proved that in the case of the Mathieu equation

$$-y'' + z(\cos 2x)y = \lambda y,$$

that is,

$$L = -(d/dx)^2, \quad By = (\cos 2x)y,$$

the SRS is irreducible [20, pp. 88–89]; see also [40]. We use Schäfke's scheme to prove that the SRS G is irreducible in the case of the simplified Jaynes–Cummings model (Theorem 3).

We focus our attention on special tri-diagonal matrices which are motivated by the analysis of second-order differential operators in the framework of Fourier method.

Example 1. Let

$$q_k = k^2, \quad b_k = c_k = k^\alpha, \quad 0 \leqslant \alpha < 2. \tag{1.3}$$

If

$$\alpha = 0, \tag{1.4}$$

we have the Mathieu matrices, and if

$$\alpha = 1/2, \tag{1.5}$$

we have the simplified Jaynes–Cumming matrices that have been considered by Boutet-de-Monvel et al. [4].

Example 2. More general q,

$$q_k = k^\gamma, \quad b_k = c_k = k^\alpha, \;\; \gamma \geqslant \alpha + 1/2.$$

The case $\gamma = 1$, $\alpha = 1/2$ comes from the Jaynes–Cumming model (see Tur [31,33]).

Example 3. The Whittaker–Hill matrices (see [18, Chapter 7] and [5])

$$q_k = k^2 \quad \text{or} \quad (2k+1)^2, \quad b_k = t - k, \quad c_k = t + k, \;\; t \geqslant 0 \text{ fixed}. \tag{1.6}$$

We do not provide details about the Fourier method or the gauge transform which lead us from the differential operator

$$-y'' + (a \cos 2x + b \cos 4x) y$$

to matrices (1.1) with (1.6). See [5,11,18,36]. In Section 7.1, Propositions 18 and 19, we use results about differential operators [37–39] to find asymptotics of the radius of analyticity R_n in the case of matrices (1.6).

Matrices (1.3)–(1.5) and (2.1), (2.2) is the main object of interest in this paper. Now we spotlight some of its results. Below $E_n(z)$ means the nth eigenvalue of $L + zB$.

Theorem 1. *Suppose* (2.1) *and* (2.2) *with* $0 \leqslant \alpha \leqslant 1/2$ *hold, and* $\lim_k b_k c_k k^{-1} = \ell$ *exists for* $\alpha = 1/2$. *Then, for* $\alpha \in [0, 1/2]$, *the regularized trace* $tr(\alpha, z)$ *is well-defined entire function, and*

$$tr(\alpha, z) \equiv \sum_1^\infty (E_n(z) - n^2) = \begin{cases} 0, & 0 \leqslant \alpha < 1/2, \\ -(\ell/2) z^2, & \alpha = 1/2. \end{cases} \tag{1.7}$$

See further comments in Section 7.6, Proposition 23.

Theorem 2. *Suppose that* (1.3) *holds and* $\alpha \in [0, 2/3]$. *For each* $R > 0$, *there is* $n_R > 0$, *such that for* $n \geqslant n_R$ *the eigenvalues* $E_n(z)$, $|z| \leqslant R$, *are well defined and*

$$E_n(z) = n^2 + z^2 \left(\frac{1-2\alpha}{2n^{2-2\alpha}} + \frac{\alpha^2 - \alpha}{n^{3-2\alpha}} + \frac{(1-2\alpha)(8\alpha^2 - 14\alpha + 3)}{24 n^{4-2\alpha}} \right) + O\left(n^{\max(2\alpha-5, 4\alpha-6)} \right). \tag{1.8}$$

See Theorem 11 in Section 4.4 also. (For $\alpha = 1/2$ similar formula was given in [4] but it was not correct).

Theorem 3. *In case* (1.3) *with* $\alpha \in [0, 0.085]$ *or* $\alpha \in [(2 - \sqrt{2})/4, 1/2]$, *the SRS*

$$G = \{(\lambda, z) \in \mathbb{C}^2 : \lambda \in Sp(L + zB)\},$$

is irreducible.

See further comments in Section 7.5, Proposition 22. In the case of anharmonic oscillator

$$Ly = -y'' + x^4 y, \quad By = x^2 y, \ \ x \in \mathbb{R}$$

a question about structure of SRS and its branching points has been raised and solved (!) by Bender and Wu [1]; see also [25–28,34,35].

The case of Mathieu–Hill operators could be deduced to Example (1.3) + (1.4); it has a longer history (see [2,3,10,19,20,37,39,40]). Some observations about Whittaker–Hill operators could be found in [5, Section 5.4].

4. In the course of proving Theorems 1–3, we observe a series of facts and inequalities about the eigenvalues of the operators $L + zB$ which could be of some interest by themselves. We discuss them in detail in related sections of the paper or in Section 7.

2. Localization of the spectra

1. Well-known methods of Perturbation Theory give information about the spectra $Sp(L + zB)$ if $L, B \in$ (1.1), (1.2). For a while, let us assume that the sequences q, b, c satisfy the conditions

$$q_k = k^2, \tag{2.1}$$

$$|b_k|, |c_k| \leqslant M k^{\alpha}, \quad 0 \leqslant \alpha < 2. \tag{2.2}$$

For each $n \in \mathbb{N}$ we set

$$\Delta_n = \{z \in \mathbb{C} : |z| \leqslant R_n\}, \quad R_n = n^{1-\alpha}/(8M). \tag{2.3}$$

Proposition 4. *Under conditions* (2.1) *and* (2.2), *the spectrum of the operator $L + zB$ is discrete, and for each n and $z \in \Delta_n$ there is exactly one eigenvalue $E_n(z)$ in the strip*

$$H_n = \{\lambda \in \mathbb{C} : n^2 - n \leqslant Re\, \lambda \leqslant n^2 + n\}.$$

Moreover, the function $E_n(z)$ is analytic in Δ_n,

$$E_n(0) = n^2 \tag{2.4}$$

and

$$\left| E_n(z) - n^2 \right| \leqslant n \quad \textit{if}\ |z| \leqslant R_n. \tag{2.5}$$

Proof. The resolvent-operator

$$R_\lambda = (\lambda - L - zB)^{-1} = R_\lambda^0 \left(1 - zBR_\lambda^0\right)^{-1} \quad \text{where } R_\lambda^0 = (\lambda - L)^{-1} \tag{2.6}$$

is well defined if

$$\lambda \notin Sp(L) \quad \text{and} \quad |z| \cdot \|BR_\lambda^0\| < 1.$$

Let K_n be the open disk with center n^2 and radius n, i.e.,

$$K_n = \{\lambda \in \mathbb{C} : |\lambda - n^2| < n\}. \tag{2.7}$$

By (2.8) (see Lemma 5 below), we have $|z| \cdot \|BR_\lambda^0\| < 1$ for $|z| \leqslant R_n$ and $\lambda \in H_n \setminus K_n$, thus

$$Sp(L + zB) \cap (H_n \setminus K_n) = \emptyset.$$

If $z = 0$, then $Sp(L) = \{k^2 : k \in \mathbb{N}\}$, so n^2 is the only eigenvalue inside the circle ∂K_n. It is simple, and for each $z \in \Delta_n$ the operator $L + zB$ has exactly one simple eigenvalue $E_n(z) \in K_n$ because

$$\dim \left(\frac{1}{2\pi i} \int_{\partial K_n} (\lambda - L - zB)^{-1} \, d\lambda \right) \equiv 1.$$

Moreover, it is well known that simple eigenvalues depend analytically on the perturbation parameter (e.g., see [15]), and therefore, for each n, $E_n(z)$ is an analytic function on Δ_n. This completes the proof of Proposition 4. □

2. The next lemma gives the estimate of the norm $\|BR_\lambda^0\|$.

Lemma 5. *Under assumptions* (2.1) *and* (2.2), *if* $\lambda = x + it \in H_n \setminus K_n$, *then*

$$\|BR_\lambda^0\| \leqslant 2M \, \max(2, 2^\alpha) n^{\alpha - 1}, \quad \forall t \in \mathbb{R}, \tag{2.8}$$

$$\|BR_\lambda^0\| \leqslant 2M \, \max(2, 2^\alpha) n^\alpha / |t| \quad \text{if } n \leqslant |t| \leqslant n^2, \tag{2.9}$$

$$\|BR_\lambda^0\| \leqslant 4M 2^\alpha |t|^{(\alpha - 2)/2} \quad \text{if } |t| \geqslant n^2. \tag{2.10}$$

Proof. Since $R_\lambda^0 = \{1/(\lambda - k^2)\}$ is a diagonal operator, while B is an off-diagonal one, the norm $\|BR_\lambda^0\|$ does not exceed, in view of (2.2),

$$\|BR_\lambda^0\| \leqslant \sup_k \frac{|b_k| + |c_{k-1}|}{|\lambda - k^2|} \leqslant \sup_k \frac{2Mk^\alpha}{|\lambda - k^2|}. \tag{2.11}$$

For every $t \in \mathbb{R}$, if $k < n$, then $|\lambda - k^2| \geqslant n - 1 \geqslant n/2$, and therefore, $k^\alpha / |\lambda - k^2| \leqslant 2n^{\alpha - 1}$. For $k = n$, we have $n^\alpha / |\lambda - n^2| \leqslant n^{\alpha - 1}$ because $|\lambda - n^2| \geqslant n$. If $n < k \leqslant 2n$, then $|\lambda - k^2| \geqslant k^2 - n^2 - n > n$, so $k^\alpha / |\lambda - k^2| \leqslant 2^\alpha n^{\alpha - 1}$; finally, if $k > 2n$ then $n < k/2$, and therefore,

$$|\lambda - k^2| \geqslant k^2 - n^2 - n \geqslant k^2 - (k/2)^2 - k/2 \geqslant k^2/2, \tag{2.12}$$

so $k^\alpha / |\lambda - k^2| \leqslant 2k^{\alpha - 2} \leqslant 2n^{\alpha - 2}$ because $\alpha < 2$. Hence (2.8) holds.

Next we consider the case where $n \leqslant |t| \leqslant n^2$. Since $|\lambda - k^2| \geqslant |t|$ we have, for $k \leqslant 2n$, that $k^\alpha / |\lambda - k^2| \leqslant (2n)^\alpha / |t|$. If $k > 2n$, then we obtain, as above, that (2.12) holds, thus

$$k^\alpha / |\lambda - k^2| \leqslant 2k^\alpha / k^2 \leqslant 2n^\alpha / n^2 \leqslant 2n^\alpha / |t|,$$

which proves (2.9).

Consider now the case where $|t| \geqslant n^2$. If $k^2 \leqslant 4|t|$, then (since $|\lambda - k^2| \geqslant |t|$)

$$k^\alpha / |\lambda - k^2| \leqslant k^\alpha / |t| \leqslant 2^\alpha |t|^{\alpha/2} / |t|.$$

If $k^2 \geqslant 4|t| \geqslant 4n^2$ then (2.12) holds, thus $k^\alpha / |\lambda - k^2| \leqslant 2k^{\alpha - 2} \leqslant 2|4t|^{(\alpha - 2)/2}$, which completes the proof of Lemma 5. □

3. By Proposition 4, for each k there is a disk Δ_k of radius $R_k(\alpha) = k^{1-\alpha}/(8M)$ with the property that the operator $L + zB$ has exactly one simple eigenvalue $E_k(z)$ in the strip H_k. If $\alpha \in [0, 1)$, then $R_k(\alpha) \uparrow \infty$ as $k \to \infty$.

Let us fix $\alpha \in [0, 1)$ and $n \in \mathbb{N}$. If $m > n$, then $\Delta_n \subset \Delta_m$, so for each $z \in \Delta_n$

$$Sp(L + zB) \cap \left(\bigcup_{m \geqslant n} H_m \right) \subset \bigcup_{m \geqslant n} K_m,$$

where K_m is defined in (2.7). Set

$$W_n = \{\lambda \in \mathbb{C} : -n < Re\,\lambda < n^2 + n,\ |Im\,\lambda| < n\}.$$

Proposition 6. *Under conditions* (2.1)–(2.3), *if* $\alpha \in [0, 1)$, *then for each* $z \in \Delta_n$

$$Sp(L + zB) \subset W_n \cup \bigcup_{m>n} K_m. \tag{2.13}$$

Moreover, the projector

$$P_*(z) = \frac{1}{2\pi i} \int_{\partial W_n} (\lambda - L - zB)^{-1}\, d\lambda \tag{2.14}$$

is well defined for $z \in \Delta_n$, *and*

$$\dim P_*(z) = n. \tag{2.15}$$

Proof. Set $H = \{\lambda \in \mathbb{C} : Re\,\lambda \leqslant n^2 + n\}$. Then,

$$\sup_k \frac{k^\alpha}{|\lambda - k^2|} = n^{\alpha-1} \quad \text{for } \lambda \in H \setminus W_n. \tag{2.16}$$

Indeed, if $k \leqslant n$, then $|\lambda - k^2| \geqslant n$, so $k^\alpha/|\lambda - k^2| \leqslant n^{\alpha-1}$; if $k > n$, then $|\lambda - k^2| \geqslant k$, thus $k^\alpha/|\lambda - k^2| \leqslant k^{\alpha-1} \leqslant n^{\alpha-1}$ because $\alpha \in [0, 1)$.

By (2.16) and (2.11), we obtain that if $|z| < n^{1-\alpha}/8M$, then

$$|z| \cdot \|BR_\lambda^0\| < 1/2 \quad \text{for } \lambda \in H \setminus W_n.$$

Therefore, in view of (2.6), for each $z \in \Delta_n$,

$$Sp(L + zB) \cap (H \setminus W_n) = \emptyset,$$

which proves (2.13) because $\mathbb{C} = H \cup \bigcup_{m>n} H_m$.

Moreover, the projector

$$P_*(z) = \frac{1}{2\pi i} \int_{\partial W_n} (\lambda - L - zB)^{-1}\, d\lambda.$$

is well defined for each $z \in \Delta_n$, and since its dimension is a constant, we obtain that $\dim P_*(z) = \dim P_*(0) = n$. □

3. The Taylor coefficients of analytic functions $E_n(z)$

1. For each $n \in \mathbb{N}$, we consider the rectangles

$$\Pi = \Pi(n, s) = \{\lambda \in \mathbb{C} : |Re(\lambda - n^2)| \leqslant n, \ |Im\,\lambda| \leqslant s\}. \tag{3.1}$$

Then, the one-dimensional Riesz projector

$$P_n(z) = \frac{1}{2\pi i} \int_{\partial\Pi} (\lambda - L - zB)^{-1}\, d\lambda \tag{3.2}$$

is well defined for $|z| \leqslant R_n$ and does not depend on s for $s > n + 1$ as it follows from (2.13) and (2.7). The integrand in (3.2) is an analytic function of $(\lambda, z) \in (H_n \setminus \Pi) \times \Delta_n$.

Since

$$E_n(z) P_n(z) = \frac{1}{2\pi i} \int_{\partial\Pi} \lambda(\lambda - L - zB)^{-1}\, d\lambda, \tag{3.3}$$

we obtain that

$$E_n(z) = Trace \left(\frac{1}{2\pi i} \int_{\partial\Pi} \lambda(\lambda - L - zB)^{-1}\, d\lambda \right). \tag{3.4}$$

Formulas (3.2)–(3.3) are basic for what follows in this section. They are used to derive formulas for the Taylor coefficients of $E_n(z)$, and to obtain a trace formula.

Let

$$E_n(z) = \sum_{k=0}^{\infty} a_k(n) z^k, \quad a_0(n) = n^2, \tag{3.5}$$

be the Taylor expansion of $E_n(z)$ at 0.

Proposition 7. *Under conditions* (2.1) *and* (2.2) *with* $\alpha \in [0, 1)$, *we have*

$$a_k(n) = \sum_j \frac{1}{2\pi i} \int_{\partial\Pi} \lambda \langle R_\lambda^0 (B R_\lambda^0)^k e_j, e_j \rangle\, d\lambda, \tag{3.6}$$

where

$$\int_{\partial\Pi} \lambda \langle R_\lambda^0 (B R_\lambda^0)^k e_j, e_j \rangle\, d\lambda = 0 \quad \textit{if } |j - n| > k,$$

$$a_k(n) = \sum_{|j-n| \leqslant k} \frac{1}{2\pi i} \int_{\partial\Pi} (\lambda - n^2) \langle R_\lambda^0 (B R_\lambda^0)^k e_j, e_j \rangle\, d\lambda, \tag{3.7}$$

$$a_k(n) \equiv 0 \quad \textit{for odd}\, k, \tag{3.8}$$

$$|a_k(n)| \leqslant 2(2k+1) \frac{(4M)^k}{n^{(1-\alpha)k-1}}, \quad k \geqslant 2. \tag{3.9}$$

Proof. By (3.2) and (2.6),

$$P_n(z) = \frac{1}{2\pi i} \int_{\partial\Pi} \sum_{k=0}^{\infty} R_\lambda^0 (B R_\lambda^0)^k z^k\, d\lambda = \sum_{k=0}^{\infty} p_k(n) z^k, \tag{3.10}$$

where the integrand-series converges absolutely and uniformly for $z \in \Delta_n$ and $\lambda \in \partial\Pi$, and

$$p_k(n) = \frac{1}{2\pi i}\int_{\partial\Pi} R_\lambda^0 (BR_\lambda^0)^k \, d\lambda, \quad k = 0, 1, 2, \ldots \tag{3.11}$$

are the Taylor coefficients of $P_n(z) \in (3.2)$.

We have

$$p_0(n)e_n = e_n, \quad p_0(n)e_j = 0 \text{ for } j \neq n. \tag{3.12}$$

Moreover, for each $k = 1, 2, \ldots$,

$$p_k(n)e_j = 0 \quad \text{if } |j - n| > k. \tag{3.13}$$

Indeed,

$$p_k(n)e_j = \frac{1}{2\pi i}\int_{\partial\Pi} R_\lambda^0 (BR_\lambda^0)^k e_j \, d\lambda. \tag{3.14}$$

Since Be_ν is a linear combination of $e_{\nu-1}$ and $e_{\nu+1}$, while $R_\lambda^0 e_\nu = 1/(\lambda - \nu^2)e_\nu$, the singularity $1/(\lambda - n^2)$ (or its power) could appear in the integrand only if $|j - n| \leqslant k$. Therefore, if $|j - n| > k$, then the integrand is an analytic function on Π, so the integral vanishes.

Since $\dim P_n(z) \equiv 1$,

$$\sum_j \langle P_n(z)e_j, e_j\rangle \equiv 1, \tag{3.15}$$

which implies, in view of (3.12) and (3.13), that

$$\sum_j \langle p_0(n)e_j, e_j\rangle = 1, \tag{3.16}$$

$$\sum_j \langle p_k(n)e_j, e_j\rangle = 0, \quad k = 1, 2, \ldots . \tag{3.17}$$

Set

$$E_n(z)\ P_n(z) = \sum_{k=0}^{\infty} d_k(n) z^k. \tag{3.18}$$

Then, by (3.5) and (3.10),

$$d_k(n) = \sum_{\nu=0}^{k} a_\nu(n) p_{k-\nu}(n). \tag{3.19}$$

Now (3.16) and (3.17) imply, in view of (3.12) and (3.13), that

$$a_k(n) = \sum_{|j-n| \leqslant k} \langle d_k(n)e_j, e_j\rangle. \tag{3.20}$$

By (3.3), taking into account the power series expansion of the resolvent, we obtain

$$a_k(n) = \sum_{|j-n| \leqslant k} \frac{1}{2\pi i}\int_{\partial\Pi} \lambda \langle R_\lambda^0 (BR_\lambda^0)^k e_j, e_j\rangle \, d\lambda. \tag{3.21}$$

Since R^0_λ is a diagonal operator, and Be_j is a linear combination of e_{j-1} and e_{j+1}, we have

$$\lambda\langle (BR^0_\lambda)^k e_j, e_j\rangle = 0, \quad \forall j \ \text{ if } k \text{ is odd.} \tag{3.22}$$

Therefore, the same argument that explains (3.13) (see (3.14) and the text after it) shows that the integrals in (3.21) are equal to zero if $|j-n| > k$, which proves (3.6).

Since Be_j is a linear combination of e_{j-1} and e_{j+1} and R^0_λ is a diagonal operator, we obtain for odd k that $(BR^0_\lambda)^k e_j$ is a finite linear combination of vectors e_ν such that $\nu - j$ is odd number, so $\nu \neq j$. Therefore, if k is odd, then for each j the integrands in (3.6) are equal to zero, which proves (3.8).

By (3.13), (3.14) and (3.17), we have

$$\sum_{|j-n|\leqslant k} \frac{1}{2\pi i}\int_{\partial\Pi} \langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j\rangle\, d\lambda = 0,$$

thus (3.6) implies (3.7).

Next we prove (3.9). Let us replace the contour $\partial\Pi$ in (3.7) by the circle $\partial K_n = \{\lambda : |\lambda - n^2| = n\}$. Fix j with $|j-n|\leqslant k$ and consider the corresponding integral. The integrand does not exceed

$$\sup_{\lambda\in\partial K_n} \left(|\lambda - n^2| \cdot \|R^0_\lambda\| \cdot \|BR^0_\lambda\|^k \right).$$

By (2.8), we have, for $\alpha \in [0, 1)$,

$$\|BR^0_\lambda\| \leqslant 4Mn^{\alpha-1} \quad \text{if } \lambda \in \partial K_n.$$

On the other hand, $|\lambda - n^2| = n$ on ∂K_n, and

$$\|R^0_\lambda\| = \sup_j \frac{1}{\lambda - j^2} \leqslant \frac{2}{n} \quad \text{for } \lambda \in \partial K_n.$$

Thus, for each j, the integrand does not exceed $2(4M)^k n^{(\alpha-1)k}$ and the length of ∂K_n is equal to $2\pi n$, which leads to the estimate (3.9). □

2. Next we give another integral representation of the coefficients $a_k(n)$.

Proposition 8. *Under conditions* (2.1) *and* (2.2) *with* $\alpha \in [0, 1)$ *we have, for each* $k \geqslant 2$,

$$a_k(1) = \varphi_k(1), \quad a_k(n) = \varphi_k(n) - \varphi_k(n-1), \ \ n \geqslant 2, \tag{3.23}$$

with

$$\varphi_k(n) = \sum_j \frac{1}{2\pi i}\int_{h_n} \lambda\langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j\rangle\, d\lambda, \tag{3.24}$$

where

$$h_n = \{\lambda \in \mathbb{C} : Re\,\lambda = n^2 + n\}, \tag{3.25}$$

and

$$\int_{h_n} \lambda\langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j\rangle\, d\lambda = 0 \quad \textit{if } |j-n| > k.$$

Moreover,

$$|\varphi_k(n)| \leqslant \frac{C_k}{n^{(1-\alpha)k-2}}, \quad k > 1, \quad C_k = (2k+1)(8M)^k. \tag{3.26}$$

Proof. Letting $s \to \infty$ in (3.6), we obtain (3.23)–(3.25). To justify this limit procedure, we have to explain that:

(i) the integrals over h_n and h_{n-1} converge;
(ii) the integrals over horizontal sides of $\partial\Pi(n, s) \in$ (3.1) go to zero as $n \to \infty$;
(iii) the integrals over h_n are equal to zero if $|j - n| > k$; and
(iv) $a_k(1) = \varphi_k(1)$.

Indeed, (i) and (ii) hold because the integrand in (3.6), for each even $k \geqslant 2$, is a linear combination of rational functions of the form

$$Q(J, \lambda) = \frac{\lambda}{(\lambda - j_0^2)(\lambda - j_1^2)\cdots(\lambda - j_k^2)}, \quad J = (j_0, \dots, j_k), \tag{3.27}$$

and therefore, the integrand decays faster than $1/|\lambda|^2$ as $|\lambda| \to \infty$.

(iii) If $j - n > k$ (respectively, $n - j > k$), then the integrand is a sum of terms (3.27) with $j_0, \dots, j_k > n$ (respectively, $j_0, \dots, j_k < n$). Consider the contour that consist of the segment $\{\lambda \in h_n : |Im\,\lambda| \leqslant s\}$ and the left half (respectively, right half) of the circle with center $n^2 + n$ and radius s. Since the integrand is an analytic function inside the contour, the integral is equal to zero. Letting $s \to \infty$ we obtain that the integral over h_n is zero, because the integral over the half-circle goes to zero due to the fact that the integrand decays as $1/|\lambda|^2$ or more rapidly.

The same argument shows, for each j, that the integral over the imaginary line $Re\,\lambda = 0$ equals zero, which explains (iv).

Finally, we prove (3.26). By (3.24), the function $\varphi_k(n)$ is a sum of at most $2k+1$ integrals over h_n of the form

$$\frac{1}{2\pi i}\int_{h_n} \lambda\langle R_\lambda^0 (BR_\lambda^0)^k e_j, e_j\rangle\, d\lambda. \tag{3.28}$$

The absolute value of integral (3.28) does not exceed

$$\frac{1}{2\pi}\int_{\mathbb{R}} F(t)\,dt \quad \text{where } F(t) = \|\lambda R_\lambda^0 (BR_\lambda^0)^k\|, \quad \lambda = n^2 + n + it. \tag{3.29}$$

Next, we estimate from above $F(t) \leqslant \|\lambda R_\lambda^0\| \cdot \|BR_\lambda^0\|^k$. Lemma 5 gives estimates of the norm $\|BR_\lambda^0\|$ on each of the three sets

$$I_1 = \{t : |t| \leqslant n\}, \quad I_1 = \{t : n \leqslant |t| \leqslant n^2\}, \quad I_1 = \{t : |t| \geqslant n^2\}.$$

On the other hand, we have

$$\|\lambda R_\lambda^0\| = \frac{|n^2 + n + it|}{|n + it|} \leqslant \begin{cases} n+1, & t \in I_1, \\ 2n^2/|t|, & t \in I_2, \\ 2, & t \in I_3. \end{cases} \tag{3.30}$$

If we combine (3.30) with estimates (2.8)–(2.10) from Lemma 5, we get

$$F(t) \leqslant \begin{cases} 2n\left(4Mn^{\alpha-1}\right)^k, & t \in I_1, \\ 2n^2\left(4Mn^{\alpha}|t|^{-1}\right)^k, & t \in I_2, \\ 2\left(8M|t|^{(\alpha-2)/2}\right)^k, & t \in I_3. \end{cases} \tag{3.31}$$

Therefore, since

$$\int_{\mathbb{R}} F(t) = \int_{I_1} F(t) + \int_{I_2} F(t) + \int_{I_3} F(t),$$

estimates (3.31) imply that (3.26) holds. □

3. Formulas (3.23) and (3.24) could be used to find the Taylor coefficients of $E_n(z)$. Indeed, under conditions (2.1) and (2.2) with $\alpha \in [0, 1)$, a computation based on the standard residue approach shows that

$$\varphi_2(n) = -\frac{b_n c_n}{2n+1}, \tag{3.32}$$

$$\varphi_4(n) = \frac{b_n^2 c_n^2}{(2n+1)^3} - \frac{b_n b_{n+1} c_n c_{n+1}}{(2n+1)^2(4n+4)} - \frac{b_n b_{n-1} c_n c_{n-1}}{4n(2n+1)^2}. \tag{3.33}$$

For any off-diagonal sequences $b, c \in (2.1) + (2.2)$, it follows from (3.26) that as $n \to \infty$

$$\varphi_k(n) \to 0 \quad \text{if } \alpha < 2/3, \quad k \geqslant 6, \tag{3.34}$$

and by (3.32), (3.33)

$$\varphi_2(n) \to 0 \ \text{ if } \alpha < 1/2, \quad \varphi_4(n) \to 0 \ \text{ if } \alpha < 3/4. \tag{3.35}$$

Now, by (3.34) and (3.23),

$$\sum_{n=1}^{\infty} a_k(n) = 0 \quad \text{if } k \geqslant 6, \quad \alpha \in [0, 1/2]. \tag{3.36}$$

If (1.3) holds, then

$$\varphi_2(n) = -\frac{n^{2\alpha}}{2n+1}, \tag{3.37}$$

$$\varphi_4(n) = \frac{n^{4\alpha}}{(2n+1)^3} - \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1)^2(4n+4)} - \frac{(n-1)^{2\alpha} n^{2\alpha}}{4n(2n+1)^2}. \tag{3.38}$$

By (3.23) and (3.37), we obtain

$$a_2(1) = \varphi_2(1) = -\frac{1}{3}, \quad a_2(n) = \frac{(n-1)^{2\alpha}}{2n-1} - \frac{n^{2\alpha}}{2n+1} \quad \text{for } n \geqslant 2. \tag{3.39}$$

Observe that $\varphi_2(n) \to 0$ if $\alpha \in [0, 1/2)$, while $\varphi_2(n) \to -1/2$ if $\alpha = 1/2$. Thus, we have

$$\sum_{n=1}^{\infty} a_2(n) = 0 \quad \text{for } \alpha \in [0, 1/2) \tag{3.40}$$

and

$$\sum_{n=1}^{\infty} a_2(n) = -\frac{1}{2} \quad \text{if } \alpha = 1/2. \tag{3.41}$$

By (3.38), we obtain that $\varphi_4(n) \to 0$ if $\alpha \in [0, 1/2]$, so (3.23) yields

$$\sum_{n=1}^{\infty} a_4(n) = 0 \quad \text{if } \alpha \in [0, 1/2]. \tag{3.42}$$

4. Asymptotics of $E_n(z)$

In this section, we study the asymptotic behavior of $E_n(z)$ for large n. Our approach is based on the fact that the eigenvalue function $E_n(z)$ satisfies a quasi-linear equation. Of course, the same estimates and formulas could be found if one follows the Raleigh–Schrödinger scheme with recurrences for the Taylor coefficients

$$\lambda(z) = \sum_{k=0}^{\infty} a_{2k}(n) z^{2k}, \quad a_0(n) = n^2,$$

$$f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad f_j \in \ell^2(\mathbb{N}), \quad f_0 = e_n,$$

as they would come if one substitute the above formulas into (4.1).

1. Throughout this section we assume that (2.1) and (2.2) with $\alpha \in [0, 1/2]$ hold, but after (4.10) we assume that (1.3) holds also.

Suppose that n and $z \in \Delta_n$ are fixed and $\lambda = E_n(z)$ is the corresponding eigenvalue of the operator $L + zB$. Then, we have

$$(L + zB) f = \lambda f \tag{4.1}$$

for some $f \neq 0$. Let P be the projector defined by $Px = \langle x, e_n \rangle e_n$, and let $Q = 1 - P$. Eq. (4.1) is equivalent to the system of two equations

$$(\lambda - L) f_1 = zPB(f_1 + f_2), \tag{4.2}$$

$$(\lambda - L) f_2 = zQB(f_1 + f_2), \tag{4.3}$$

where $f_1 = Pf, \; f_2 = Qf$. The operator $\lambda - L$ is invertible on the range of the projector Q, we set

$$De_k = \frac{1}{\lambda - k^2} e_k \;\text{ if } k \neq n, \quad De_n = 0. \tag{4.4}$$

Then, D is well defined in ℓ^2, and $(\lambda - L)Dx = x$ on the range of Q.

Acting on both sides of (4.3) by the operator BD, we obtain

$$Bf_2 = zTBf_1 + zTBf_2, \tag{4.5}$$

where

$$T = BD. \tag{4.6}$$

The operator $1 - zT$ is invertible for each $z \in \Delta_n$. Indeed, since $Te_k = BR^0_\lambda e_k$ for $k \neq n$ and $Te_n = 0$, the proof of (2.8) shows that

$$\|T\| \leqslant 4M \cdot n^{\alpha-1} \quad \text{for } \lambda \in H_n. \tag{4.7}$$

Thus, we have

$$\|zT\| \leqslant |z| \cdot \|T\| < 1$$

for each $z \in \Delta_n$ and each $\lambda \in H_n$.

Solving (4.5) for Bf_2, we obtain

$$Bf_2 = z(1 - zT)^{-1} TBf_1. \tag{4.8}$$

Inserted into (4.2), this leads to

$$(\lambda - L)f_1 = zPBf_1 + z^2 P(1 - zT)^{-1} TBf_1,$$

which implies (since $1 + zT(1 - zT)^{-1} = (1 - zT)^{-1}$)

$$(\lambda - L)f_1 = zP(1 - zT)^{-1} Bf_1, \tag{4.9}$$

where $f_1 = const \cdot e_n \neq 0$ (otherwise, by (4.3) it follows that $f_2 = 0$, so $f = f_1 + f_2 = 0$, which contradicts $f \neq 0$). Since $Le_n = n^2 e_n$, Eq. (4.9) is equivalent to

$$\lambda - n^2 = z\langle (1 - zT)^{-1} Be_n, e_n\rangle. \tag{4.10}$$

Since

$$Be_k = (k-1)^\alpha e_{k-1} + k^\alpha e_{k+1}, \tag{4.11}$$

we have, by (4.4) and (4.6), that

$$Te_k = \frac{1}{\lambda - k^2}\left((k-1)^\alpha e_{k-1} + k^\alpha e_{k+1}\right), \quad Te_n = 0, \tag{4.12}$$

and therefore,

$$\langle T^{2k} Be_n, e_n\rangle = 0, \quad k = 0, 1, 2, \ldots. \tag{4.13}$$

Let

$$E_n(z) = n^2 + a_1(n)z + a_2(n)z^2 + \cdots$$

be the Taylor expansion of $E_n(z)$. Set for convenience

$$\zeta(z) = E_n(z) - n^2 = a_1(n)z + a_2(n)z^2 + \cdots. \tag{4.14}$$

Then, by (4.10),

$$\zeta(z) = \langle TBe_n, e_n\rangle z^2 + \langle T^3 Be_n, e_n\rangle z^4 + \langle T^5 Be_n, e_n\rangle z^6 + \cdots, \tag{4.15}$$

where, by (4.12), the operator T depends rationally on $\lambda = E_n(z) = \zeta(z) + n^2$. It is easy to see, by induction, that (4.15) yields $a_{2k+1}(n) = 0,\ k \in \mathbb{N}$ (in fact, we know this from Section 3.1, see (3.8)). Thus, we have

$$\zeta(z) = a_2(n)z^2 + a_4(n)z^4 + \cdots. \tag{4.16}$$

One may use (4.15) to compute the Taylor coefficients of $\zeta(z)$. Indeed, the right side of (4.15) is a power series in z which coefficients are rational functions of $\lambda = \zeta + n^2$ without a singularity at 0. So, replacing these rational functions with their power series expansions at 0, and replacing ζ with its power expansion (4.14), we obtain (comparing the resulting power series expansion on the left and on the right) a system of equations for the coefficients $a_2(n), a_4(n), \ldots$.

Next we compute some of these coefficients. By (4.11)–(4.15), it follows that

$$\begin{aligned}\zeta = {} & z^2\left(\frac{(n-1)^{2\alpha}}{2n-1+\zeta} - \frac{n^{2\alpha}}{2n+1-\zeta}\right) \\ & + z^4\left(\frac{(n-1)^{2\alpha}(n-2)^{2\alpha}}{(2n-1+\zeta)^2(4n-4+\zeta)} - \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1-\zeta)^2(4n+4+\zeta)}\right) + \cdots \\ = {} & z^2\left(\frac{(n-1)^{2\alpha}}{2n-1}\left(1 - \frac{\zeta}{2n-1} + \cdots\right) - \frac{n^{2\alpha}}{2n+1}\left(1 + \frac{\zeta}{2n+1} + \cdots\right)\right) \\ & + z^4\left[\frac{(n-1)^{2\alpha}(n-2)^{2\alpha}}{(2n-1)^2(4n-4)}\left(1 - \frac{\zeta}{2n-1} + \cdots\right)^2\left(1 - \frac{\zeta}{4n-4} + \cdots\right)\right. \\ & \left. - \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1)^2(4n+4)}\left(1 + \frac{\zeta}{2n+1} + \cdots\right)^2\left(1 + \frac{\zeta}{4n+4} + \cdots\right)\right] + \cdots . \end{aligned} \tag{4.17}$$

Hence, we obtain

$$a_2(\alpha, n) = \frac{(n-1)^{2\alpha}}{2n-1} - \frac{n^{2\alpha}}{2n+1}, \quad n \geqslant 2, \tag{4.18}$$

$$\begin{aligned} a_4(\alpha, n) = {} & (-a_2(n))\left(\frac{(n-1)^{2\alpha}}{(2n-1)^2} + \frac{n^{2\alpha}}{(2n+1)^2}\right) \\ & + \frac{(n-1)^{2\alpha}(n-2)^{2\alpha}}{(2n-1)^2(4n-4)} - \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1)^2(4n+4)}, \quad n \geqslant 3. \end{aligned} \tag{4.19}$$

The same method gives

$$a_6(\alpha, n) = \sigma_1(n) - a_2(n)\sigma_2(n) - a_4(\alpha, n)\sigma_3(n), \quad n \geqslant 4, \tag{4.20}$$

where

$$\begin{aligned} \sigma_1(n) = {} & \frac{(n-1)^{2\alpha}(n-2)^{4\alpha}}{(2n-1)^3(4n-4)^2} - \frac{n^{2\alpha}(n+1)^{4\alpha}}{(2n+1)^3(4n+4)^2} \\ & + \frac{(n-1)^{2\alpha}(n-2)^{2\alpha}(n-3)^{2\alpha}}{(2n-1)^2(4n-4)^2(6n-9)} - \frac{n^{2\alpha}(n+1)^{2\alpha}(n+2)^{2\alpha}}{(2n+1)^2(4n+4)^2(6n+9)}, \end{aligned} \tag{4.21}$$

$$\begin{aligned} \sigma_2(n) = {} & \frac{(n-1)^{2\alpha}(n-2)^{2\alpha}}{(2n-1)^2(4n-4)}\left(\frac{2}{2n-1} + \frac{1}{4n-4}\right) \\ & + \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1)^2(4n+4)}\left(\frac{2}{2n+1} + \frac{1}{4n+4}\right) + \frac{n^{2\alpha}}{(2n+1)^3} - \frac{(n-1)^{2\alpha}}{(2n-1)^3}, \end{aligned} \tag{4.22}$$

$$\sigma_3(n) = \frac{(n-1)^{2\alpha}}{(2n-1)^2} + \frac{n^{2\alpha}}{(2n+1)^2}. \tag{4.23}$$

Of course, the case of small n requires a special treatment. For example, if $n = 1$, then with

$$\zeta = a_2(1)z^2 + a_4(1)z^4 + a_6(1)z^6 + \cdots,$$

we have

$$\zeta = z^2\left(\frac{1}{\zeta-3}\right) + z^4\left(\frac{2^{2\alpha}}{(\zeta-3)^2(\zeta-8)}\right)$$
$$+ z^6\left(\frac{2^{4\alpha}}{(\zeta-3)^3(\zeta-8)^2} + \frac{2^{2\alpha}3^{2\alpha}}{(\zeta-3)^2(\zeta-8)^2(\zeta-15)}\right) + \cdots,$$

which leads to

$$a_2(1) = -\frac{1}{3}, \quad a_4(1) = \frac{1}{27} - \frac{2^{2\alpha}}{72} \tag{4.24}$$

(compare with (3.23), (3.37), (3.38)), and

$$a_6(1) = -\frac{2^{4\alpha}}{3^3\cdot 8^2} - \frac{2^{2\alpha}3^{2\alpha}}{3^3\cdot 8^2\cdot 5} + \frac{2^{2\alpha}}{3\cdot 8^2} - \frac{2}{3^5}. \tag{4.25}$$

2. The following lemma gives the asymptotic behavior of $a_{2k}(\alpha, n)$ as $n \to \infty$.

Lemma 9. *Under condition* (1.3), *if* $\alpha \in [0, 1)$, *then*

$$a_{2k}(\alpha, n) = O(n^{2k(\alpha-1)}). \tag{4.26}$$

Proof. We prove (4.26) by induction in k. If $k = 1$, then (4.18) yields

$$a_2(\alpha, n) = O(n^{2(\alpha-1)}). \tag{4.27}$$

If $k = 2$, then (4.19) gives $a_4(\alpha, n)$ as a sum of two expressions. For the first one we obtain, in view of (4.27),

$$a_2(\alpha, n)\left(\frac{(n-1)^{2\alpha}}{(2n-1)^2} + \frac{n^{2\alpha}}{(2n+1)^2}\right) = O(n^{2(\alpha-1)})\cdot O(n^{2(\alpha-1)}) = O(n^{4(\alpha-1)}).$$

The remaining part of (4.19) is

$$\frac{(n-1)^{2\alpha}(n-2)^{2\alpha}}{(2n-1)^2(4n-4)} - \frac{n^{2\alpha}(n+1)^{2\alpha}}{(2n+1)^2(4n+4)}. \tag{4.28}$$

Each term of this difference is $O(n^{4\alpha-3})$. But (4.28) is $O(n^{4(\alpha-1)})$ due to the Mean Value Theorem. Indeed, let $f(m) = m^{2\alpha}(m+1)^{2\alpha}(4m+4)^{-1}$ and $g(m) = (2m+1)^{-2}$. Then, (4.28) may be written as

$$f(n-2)g(n-1) - f(n)g(n) = (f(n-2) - f(n))g(n-1)$$
$$+ f(n)(g(n-1) - g(n)).$$

Since

$$f'(t) = O(f(n)/n), \quad g'(t) = O(g(n)/n) \text{ for } t \in [n-2, n],$$

by the Mean Value Theorem expression (4.28) is $O(n^{4\alpha-4})$, which proves (4.26) for $k = 2$.

Fix $k \geqslant 3$ and assume that (4.26) holds for $1, \ldots, k-1$. Then, by (4.15) and (4.16) we obtain, in view of (4.17),

$$a_{2k} = \langle T^{2k-1}Be_n, e_n\rangle + \sum C_{m_1\ldots m_{k-1}} a_2^{m_1}\cdots a_{2(k-1)}^{m_{k-1}},$$

where $m_1 + 2m_2 + \cdots + (k-1)m_{k-1} = k$, and

$$T = BD \text{ with } De_n = 0, \quad De_\nu = \frac{1}{n^2 - \nu^2} e_\nu.$$

In addition, for each term of the sum, we have

$$C_{m_1 \ldots m_{k-1}} a_2^{m_1} \cdots a_{2(k-1)}^{m_{k-1}} = O(n^{2k(\alpha-1)}).$$

(See (4.20)–(4.23) for the case $k = 3$.) Thus, Lemma 9 will be proved if we show that

$$\langle T^{2k-1} B e_n, e_n \rangle = O(n^{2k(\alpha-1)}). \tag{4.29}$$

Set

$$B = B_{+1} + B_{-1} \quad \text{and} \quad T = T_{+1} + T_{-1}, \tag{4.30}$$

where

$$B_{+1} e_k = k^\alpha e_{k+1}, \quad B_{-1} e_k = (k-1)^\alpha e_{k-1}, \tag{4.31}$$

and

$$T_{+1} = B_{+1} D, \quad T_{-1} = B_{-1} D. \tag{4.32}$$

Then

$$\langle T^{2k-1} B e_n, e_n \rangle = \sum_\varepsilon \omega(\varepsilon), \tag{4.33}$$

where the summation is over all $2k$-tuples, $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_{2k})$ with $\varepsilon_\nu = \pm 1$, and

$$\omega(\varepsilon) = \langle T_{\varepsilon_{2k-1}} \cdots T_{\varepsilon_2} B_{\varepsilon_1} e_n, e_n \rangle. \tag{4.34}$$

Let

$$\delta(\varepsilon) = (\delta_1, \ldots, \delta_{2k}), \quad \delta_\nu = \delta_\nu(\varepsilon) = \varepsilon_1 + \cdots + \varepsilon_\nu, \quad \nu = 1, \ldots, 2k; \tag{4.35}$$

then, $T_{\varepsilon_\nu} \cdots T_{\varepsilon_2} B_{\varepsilon_1} e_n = const \cdot e_{n+\delta_\nu}$. Therefore, since $De_n = 0$, we have $\omega(\varepsilon) \neq 0$ if and only if $\delta_{2k} = 0$ and $\delta_\nu \neq 0$ for $\nu \neq 2k$.

Now (4.33) implies that

$$\langle T^{2k-1} B e_n, e_n \rangle = \sum_{\varepsilon \in e^+} [\omega(\varepsilon) + \omega(-\varepsilon)], \tag{4.36}$$

where the summation is over the set e^+ of all $2k$-tuples ε, such that $\delta_n u(\varepsilon) > 0$ for $\nu = 1, \ldots, 2k-1$. Since the cardinality of e^+ does not exceed 2^{2k}, (4.29) will be proved if we show, for each $\varepsilon \in e^+$, that

$$\omega(\varepsilon) + \omega(-\varepsilon) = O(n^{2k(\alpha-1)}). \tag{4.37}$$

By (4.30)–(4.32), we obtain

$$\omega(\varepsilon) = -\frac{\prod_{\nu=1}^{2k} (n + \delta_{\nu-1} + (\varepsilon_\nu - 1)/2)^\alpha}{\prod_{\nu=1}^{2k-1} (\delta_\nu (2n + \delta_\nu))}, \quad \delta_0 = 0, \quad \delta_\nu = \varepsilon_1 + \cdots + \varepsilon_\nu.$$

Now, as above, the Mean Value Theorem may be used to show that (4.37) holds. This completes the proof of Lemma 9. □

3. *Proof of Theorem* 2: By Proposition 4 we know that, with $R_n = n^{1-\alpha}/(8M)$,

$$|E_n(z) - n^2| \leqslant n \quad \text{for } z \in \Delta_n = \{\zeta : |\zeta| \leqslant R_n\}.$$

Lemma 10. *For each* $k = 1, 2, \ldots,$

$$|a_k(n)| = \frac{1}{k!}\left|E_n^{(k)}(0)\right| \leqslant (8M)^k n^{1-k(1-\alpha)}. \tag{4.38}$$

Proof. Indeed, $E_n(z)$ is analytic in Δ_n. Therefore, the Cauchy inequality for the Taylor coefficients of $E_n(z)$ at 0 gives (4.38). $\square$

Now, for $|z| \leqslant R$, we obtain

$$\left|E_n(z) - n^2 - \sum_{k=1}^{6} a_{2k} z^{2k}\right| \leqslant \sum_{k=7}^{\infty} |a_{2k}(n)| R^{2k} \leqslant \frac{C_n}{n^{13-14\alpha}}, \tag{4.39}$$

where $C_n = (8MR)^{14} \sum_{k \geqslant 0} (8MR)^{2k}/n^{2k(1-\alpha)}$ is a bounded sequence.

On the other hand, (4.18) and (4.19) imply that

$$a_2(\alpha, n) = \frac{(1-2\alpha)}{2n^{2-2\alpha}} + \frac{(\alpha^2 - \alpha)}{n^{3-2\alpha}} + \frac{(1-2\alpha)(8\alpha^2 - 14\alpha + 3)}{24 n^{4-2\alpha}} + O(n^{2\alpha-5}) \tag{4.40}$$

and

$$a_4(\alpha, n) = O(n^{4\alpha-6}). \tag{4.41}$$

Formulas (4.20)–(4.23) yield

$$a_6(\alpha, n) = O(n^{6\alpha-10}). \tag{4.42}$$

Analogous computations show that

$$a_8(\alpha, n) = O(n^{8\alpha-14}). \tag{4.43}$$

Finally, by Lemma 9 we obtain

$$a_{10}(\alpha, n) = O(n^{10\alpha-10}), \quad a_{12}(\alpha, n) = O(n^{12\alpha-12}). \tag{4.44}$$

Now (4.39)–(4.44) imply (1.8). Indeed, if $\alpha \in [0, 1/2]$, then $2\alpha - 5 \geqslant 4\alpha - 6$; moreover,

$$12\alpha - 12 \leqslant 10\alpha - 10 \leqslant 2\alpha - 5$$

and $14\alpha - 13 \leqslant 2\alpha - 5$, thus (1.8) holds.

If $\alpha \in [1/2, 2/3]$, then $2\alpha - 5 \leqslant 4\alpha - 6$; so, since

$$12\alpha - 12 \leqslant 10\alpha - 10 \leqslant 4\alpha - 6$$

and $14\alpha - 13 \leqslant 4\alpha - 6$, we obtain that (1.8) holds. This completes the proof of Theorem 2. $\square$

4. We consider separately the case where $\alpha = 1/2$ in the following theorem.

Theorem 11. *If* $|z| \leqslant R$, *then*

$$E_n(1/2, z) = n^2 - \frac{z^2}{4n^2} - \frac{2z^2 + 3z^4}{32n^4} + O(1/n^6). \tag{4.45}$$

Proof. If $\alpha = 1/2$, then (4.39) implies

$$\left| E_n(z) - n^2 - \sum_{k=1}^{6} a_{2k} z^{2k} \right| \leqslant \sum_{k=7}^{\infty} |a_{2k}(n)| R^{2k} \leqslant \frac{C_n}{n^6}, \tag{4.46}$$

where $C_n = (8MR)^{14} \sum_{k \geqslant 0} (8MR)^{2k}/n^k$ is a bounded sequence. On the other hand, from (4.18) to (4.23), it follows that

$$a_2 = -\frac{1}{4n^2 - 1}, \tag{4.47}$$

$$a_4 = \frac{1}{4(2n+1)^3} - \frac{1}{4(2n-1)^3} \tag{4.48}$$

$$a_6 = -\frac{1}{(2n+3)(2n+1)^5(2n-1)} + \frac{1}{(2n+1)(2n-1)^5(2n-3)}. \tag{4.49}$$

The same approach that leads to (4.18)–(4.23) gives

$$a_8 = \frac{-327 - 16\,080n^2 - 63\,136n^4 + 29\,440n^6 + 39\,168n^8}{32(n-1)(n+1)(2n-3)(2n+3)(2n-1)^7(2n+1)^7}, \tag{4.50}$$

and

$$a_{10} = \frac{3915 + 280\,676n^2 + 2\,496\,992n^4 + 2\,635\,904n^6 - 3\,111\,168n^8 - 1\,158\,144n^{10}}{8(n-1)(n+1)(2n-3)(2n+3)(2n-5)(2n+5)(2n-1)^9(2n+1)^9}. \tag{4.51}$$

By (4.48)–(4.51) we obtain

$$a_2(1/2, n) = -\frac{1}{4n^2} - \frac{1}{16n^4} + O(n^{-6}), \tag{4.52}$$

$$a_4(1/2, n) = -\frac{3}{32n^4} + O(n^{-6}), \tag{4.53}$$

and

$$a_6(1/2, n) = O(n^{-8}), \quad a_8(1/2, n) = O(n^{-10}), \quad a_{10}(1/2, n) = O(n^{-14}). \tag{4.54}$$

In addition, Lemma 9 implies that

$$a_{12}(1/2, n) = O(n^{-6}). \tag{4.55}$$

Now (4.45) follows from (4.46) and (4.52)–(4.55). □

Remark 12. We evaluate a_{12} in (4.55) by using the general estimate (4.26) from Lemma 9. However, *a direct computation of the coefficients $a_{2k}(n)$ for $6 \leqslant k \leqslant 14$ shows that each of them is $O(1/n^{16})$*. Estimating the remainder as in the proof of Theorem 11, we get

$$\sum_{k \geqslant 15} a_{2k}(n) z^k = O(1/n^{14}), \quad |z| \leqslant R. \tag{4.56}$$

So, by (4.51), we have

$$E_n(z) = n^2 + a_2 z^2 + a_4(n) z^4 + a_6(n) z^6 + a_8(n) z^8 + O(1/n^{14}), \quad |z| \leqslant R. \tag{4.57}$$

It follows from here, in view of (4.47)–(4.50), that

$$E_n(z) = n^2 + \sum_{k=1}^{6} P_k(z) \frac{1}{n^{2k}} + O(1/n^{14}), \tag{4.58}$$

where

$$\begin{aligned} &P_1(z) = -\frac{z^2}{4}, \quad P_2(z) = -\frac{2z^2 + 3z^4}{32}, \quad P_3(z) = -\frac{z^2 + 5z^4}{64}, \\ &P_4(z) = \frac{-2z^2 - 21z^4 + 28z^6}{512}, \quad P_5(z) = \frac{-8z^2 - 144z^4 + 1920z^6 + 153z^8}{8192}, \\ &P_6(z) = \frac{-2z^2 - 55z^4 + 5192z^6 + 880z^8}{8192}. \end{aligned} \tag{4.59}$$

See further discussion in Section 7.3.

5. Analytic continuation of eigenvalues and regularized trace

1. Each eigenvalue $E_k(z)$, as we have seen in Proposition 4, is well defined and simple if $|z| \leqslant R_k = k^{1-\alpha}/8M$. We are going to show that it is possible to continue $E_k(z)$ analytically as z is moving along a smooth curve which goes around singular points $\zeta \in S$, where S is a countable set without a finite point of accumulation.

Fix $n \in \mathbb{N}$ and consider the rectangle

$$W \equiv W_n = \{\lambda \in \mathbb{C} : -n < Re\,\lambda < n^2 + n, \ |Im\,\lambda| < n\}.$$

By Proposition 6, the projector

$$P_*(z) = \frac{1}{2\pi i} \int_{\partial W} (\lambda - L - zB)^{-1} \, d\lambda$$

is well defined for $z \in \Delta_n$ and

$$\dim P_*(z) = n.$$

Consider the analytic functions

$$\sigma_j(z) = Trace\left(\frac{1}{2\pi i} \int_{\partial W} \lambda^j (\lambda - L - zB)^{-1} \, d\lambda \right), \quad 1 \leqslant j \leqslant n. \tag{5.1}$$

If $|z|$ is small, say $|z| < \varepsilon < R_1$, then

$$\sigma_j(z) = \sum_{k=1}^{n} (E_k(z))^j, \quad 1 \leqslant j \leqslant n, \tag{5.2}$$

where all $E_k(z)$ are well defined. Moreover,

$$\prod_1^n (\lambda - E_k) = \sum_0^n Q_{n-j}(E)\lambda^j, \tag{5.3}$$

where $\{Q_i\}_1^n$, $Q_0 \equiv 1$, are symmetric polynomials of $\{E_k\}_1^n$. But $\{\sigma_j\}_1^n$ is a basis system of symmetric polynomials (see, e.g. [17]), and therefore,

$$Q_j = q_j(\sigma) \tag{5.4}$$

are polynomials of σ's. Thus,

$$\prod_1^n (\lambda - E_k) = \sum_0^n q_{n-j}(\sigma(z))\lambda^j, \tag{5.5}$$

at least for small $|z|$, say $|z| < \varepsilon$. However, the coefficients $c_j(z) = q_{n-j}(\sigma(z))$ are well defined by (5.4), (5.1) in the entire disk Δ_n and analytic there. Factorization (5.3) becomes

$$\prod_1^n (\lambda - E_k(z)) = \sum_0^n c_j(z)\lambda^j, \tag{5.6}$$

and the equation

$$c(z, \lambda) := \sum_0^n c_j(z)\lambda^j = 0, \quad |z| \leqslant R_n, \tag{5.7}$$

defines over Δ_n the surface

$$G_n = \{(\lambda, z) \in \mathbb{C} \times \Delta_n : c(z, \lambda) = 0\} \tag{5.8}$$

with n sheets and possible branching points z_* if the polynomial $\sum_0^n c_j(z_*)\lambda^j$ has multiple roots. Such a point z_* is a root of the resultant

$$r(z) = R(c(z, \cdot), c'_\lambda(z, \cdot)) \tag{5.9}$$

of the polynomial $c(z, \lambda)$ and its derivative c'_λ. Notice that $r(z)$ is an analytic function of z, $|z| \leqslant R_n$, because the resultant is a polynomial of $c_j(z) \in$(5.7). If $z = 0$, then

$$\sum_0^n c_j(0)\lambda^j = \prod_1^n (\lambda - k^2), \tag{5.10}$$

and all zeros are simple. Therefore, $r(0) \neq 0$, so the resultant $r(z)$ is not identically zero. Thus, the set

$$\Sigma_n = \{z \in \Delta_n : r(z) = 0\} \tag{5.11}$$

is finite. By Proposition 4 we can conclude that

$$\Sigma_n \subset \Sigma_{n+1} \quad \text{and} \quad \Sigma_{n+1} \cap \Delta_n = \Sigma_n. \tag{5.12}$$

Thus, the set

$$S = \bigcup \Sigma_n \tag{5.13}$$

is countable and has no finite points of accumulation.

We have proved the following.

Proposition 13. *Under the conditions of Proposition* 4, *there is a countable set S without finite accumulation points*, *such that if*

$$\gamma = \{z(t) : 0 \leqslant t \leqslant T\}, \quad z(0) = 0, \ \ \gamma \cap S = \emptyset$$

is a smooth curve, *then each eigenvalue function $E_k(z)$, $E_k(0) = k^2$*, *can be extended analytically along the curve γ*.

2. We define *SRS* of the pair (L, B) as

$$G = \{(\lambda, z) \in \mathbb{C}^2 : (L + zB)f = \lambda f, \ f \in \ell^2(\mathbb{N}), \ f \neq 0\}. \tag{5.14}$$

Proposition 14. *Under the conditions of Proposition* 4, *for each $z \notin S$ the surface G has infinitely many sheets over a neighborhood $U_\varepsilon \ni z$ for small enough $\varepsilon(z) > 0$. Each branching point $z_* \in S$ is of finite order*.

Proof. Everything has been already explained. The surface G over Δ_n is defined by (5.7), and, by (5.7)–(5.11), $\lambda(z)$ has branching points $z_* \in \Delta_n$ of order $\leqslant n$. □

3. We follow the 1975 Schäfke construction (see [20, pp. 88–89]), as it is presented by Volkmer [40], to analyze whether the SRS G is *irreducible*.

Let $k, j \in \mathbb{N}$. We call k and j *equivalent*, $k \sim j$, if there is a smooth curve

$$\varphi : [0, T] \to \mathbb{C} \setminus S, \quad \varphi(0) = \varphi(T) = 0,$$

such that the analytic continuation of $E_k(z)$ along φ leads to $E_j(z)$. (An SRS G is *irreducible* if $\mathbb{N}$ is the only equivalence class, i.e., $k \sim j$ for any $k, j \in \mathbb{N}$.)

Such construction, carried for each $k \in \mathbb{N}$, defines a mapping

$$\pi_\varphi : \mathbb{N} \to \mathbb{N}, \quad \pi(k) = j,$$

such that $\pi_{\varphi^{-1}}(j) = k$, where $\varphi^{-1}(t) = \varphi(T - t)$. With $R_n \to \infty$, we have for some n that $\max_{[0,T]} |\varphi(t)| \leqslant R_n$. Therefore, by Proposition 4,

$$\pi_\varphi(k) = k \quad \text{if } k > n. \tag{5.15}$$

Lemma 15. *Let $\mathcal{M}$ be an equivalence class* (*or union of equivalence classes*), *and $n \in \mathbb{N}$. Then*, *the function*

$$\widetilde{E}_n(z) = \sum_{k \in \mathcal{M}, k \leqslant n} E_k(z) \tag{5.16}$$

(*which is well defined and analytic for small enough $|z|$*) *can be extended analytically on the disk $\{z : |z| \leqslant R_n\}$, $R_n = n^{1-\alpha}/(8M)$*.

Proof. Take any smooth curve $\varphi : [0, T] \to \Delta_n \setminus S$, such that $\varphi(0) = \varphi(T) = 0$. Then, $\pi = \pi_\varphi : \mathcal{M} \to \mathcal{M}$ is a bijection, and (5.15) holds, so π permutes the finite set $\{k \in \mathcal{M} : k \leqslant n\}$. Therefore, $\widetilde{E}_n(z)$ can be continued analytically, term by term in (5.16), and the result will be

$$\sum_{k\in\mathcal{M},k\leqslant n} E_{\pi(k)}(z) = \sum_{j\in\mathcal{M},j\leqslant n} E_j(z) = \widetilde{E}_n(z), \quad z \in \Delta_n \setminus S,$$

i.e., the same function. By Proposition 6, if $|z| \leqslant R_n$, then we have exactly n eigenvalues on the left of the line $h_n = \{Re\,\lambda = n^2 + n\}$, and all of them lie in the rectangle W_n. Therefore,

$$\left|\widetilde{E}_n(z)\right| \leqslant n(n^2 + 2n). \tag{5.17}$$

So, the function $\widetilde{E}_n(z)$ is analytic and bounded on $\Delta_n \setminus S$, while the set $\Delta_n \cap S$ is finite. Thus, it is analytic in the disk Δ_n. $\square$

Inequality (5.17) cannot be improved essentially because

$$\sum_1^n E_k(0) = \sum_1^n k^2 = n(n+1)(2n+1)/6 \sim n^3.$$

However, we can regularize $\tilde{E}_k(z)$ by considering $\tilde{E}_k(z) - \tilde{E}_k(0)$, where $\tilde{E}_k(0)$ is real.

Again by Proposition 6, if $|z| \leqslant R_n$, then the operator $L + zB$ has n eigenvalues that lie in the rectangle W_n, so the absolute value of the imaginary part of each of these eigenvalues is less than n. Therefore,

$$\left|Im\left(\widetilde{E}_n(z) - \widetilde{E}_n(0)\right)\right| \leqslant n^2. \tag{5.18}$$

By Borel–Caratheodory theorem (see Titchmarsh [30, Chapter 5, 5.5 and 5.51]), if $g(z)$ is analytic in the disk $|z| < R$, $g(0) = 0$ and $|Im\, g(z)| \leqslant C$, then $|g(z)| \leqslant 2C$ for $|z| \leqslant R/2$. Thus, (5.18) implies

$$\left|\widetilde{E}_n(z) - \widetilde{E}_n(0)\right| \leqslant 2n^2 \quad \text{for } |z| \leqslant R_n/2. \tag{5.19}$$

This conclusion is valid for each equivalence class, or union of equivalence classes $\mathcal{M}$; in particular, for $\mathcal{M} = \mathbb{N}$.

4. *Definition of the regularized trace* $tr(z)$: Now we are ready to define an entire function $tr(z)$, the regularized trace of $L + zB$, under conditions (2.1) and (2.2) with $\alpha < 1/2$, or (1.3) with $\alpha = 1/2$.

For small z, $|z| \leqslant R_1 = 1/(8M)$, all $E_n(z)$ are well defined, and

$$\begin{aligned} tr(z) &= \sum_{n=1}^\infty \left(E_n(z) - n^2\right) = \sum_{n=1}^\infty \left(\sum_{k=1}^\infty a_{2k}(n) z^{2k}\right) \\ &= \sum_{n=1}^\infty \left(a_2(n)z^2 + a_4(n)z^4 + \sum_{k=3}^\infty a_{2k}(n)z^{2k}\right) \\ &= z^2 \cdot \lim_{p\to\infty} \varphi_2(p) + z^4 \cdot \lim_{p\to\infty} \varphi_4(p) + \sum_{n=1}^\infty \sum_3^\infty \cdots . \end{aligned} \tag{5.20}$$

By (3.32)–(3.35), the latter limits are well defined and the third term is an absolutely convergent series. Indeed, by (4.38), Lemma 10, we have for $|z| < 1/(8M)$ that

$$\sum_{n=1}^{\infty}\left(\sum_{k=3}^{\infty}|a_{2k}(n)|\right)|z|^{2k} \leqslant \sum_{n=1}^{\infty}\left(\sum_{k=3}^{\infty}\frac{(8M)^{2k}}{n^{2k(1-\alpha)-1}}\right)\frac{1}{(8M)^{2k}}$$
$$= \sum_{n=1}^{\infty}\frac{n}{n^{6(1-\alpha)}}\cdot\frac{1}{1-n^{\alpha-1}} < \infty \quad \text{for } \alpha < 2/3.$$

Therefore, (5.20) defines $tr(z)$ as an analytic function in the disk $|z|\leqslant 1/(8M)$.

Fix $N \in \mathbb{N}$ and consider the analytic function $\tilde{E}_N(z),\ z \in \Delta_N$, given by Lemma 15 in the case where $\mathcal{M} = \mathbb{N}$. For small $|z|$, we have

$$tr(z) = \widetilde{E}_N(z) - \widetilde{E}_N(0) + \sum_{n=N+1}^{\infty}\left(E_n(z) - n^2\right). \tag{5.21}$$

The same formula gives the analytic extension of $tr(z)$ on Δ_N because the series on the right side of (5.21) converges uniformly on Δ_N. Indeed, with $E_n(z) = a_2(n)z^2 + a_4(n)z^4 + \cdots$, we have

$$\sum_{N+1}^{\infty}\left(E_n(z) - n^2\right) = \left(\sum_{N+1}^{\infty} a_2(n)\right)z^2 + \left(\sum_{N+1}^{\infty} a_4(n)\right)z^4$$
$$+ \sum_{n=N+1}^{\infty}\sum_{k=3}^{\infty} a_{2k}(n)z^{2k}. \tag{5.22}$$

By (3.9) we obtain, for $n \geqslant N+1$ and $|z|\leqslant R_N = N^{1-\alpha}/(8M)$,

$$\sum_{k=3}^{\infty}|a_{2k}(n)|\,|z|^{2k} \leqslant \sum_{k=6}^{\infty}\frac{4k+2}{n^{(1-\alpha)k-1}}(4M)^k R_N^k \leqslant C(N,\alpha)\left(\frac{1}{n}\right)^{6(1-\alpha)-1}, \tag{5.23}$$

where $C(N,\alpha) = N^{6(1-\alpha)}\sum_{k\geqslant 6}(4k+2)2^{-k} < \infty$. Now, in view of (5.22), estimate (5.23) implies that the series in (5.21) converges uniformly in Δ_N if $\alpha \in [0, 1/2]$, thus $tr(z)$ can be extended analytically in the disk Δ_N. Since $\cup_N \Delta_N = \mathbb{C}$ *this defines $tr(z)$ as an entire function.*

5. *Proof of Theorem* 1: According to the previous subsection, $tr(z)$ is an entire function. Therefore, it is enough to prove (1.7) only for small $|z|$, or to evaluate its Taylor coefficients. By (5.22) $tr(z) = \sum_1^\infty A_{2k}z^{2k}$, where

$$A_{2k} = \sum_{n=1}^{\infty} a_{2k}(n) = \lim_{p\to\infty}\varphi_{2k}(p).$$

If $\alpha < 1/2$, then we have, by (3.34) and (3.35),

$$\lim_{p\to\infty}\varphi_{2k}(p) = 0, \quad k = 1, 2, \ldots,$$

and therefore, $tr(z) \equiv 0$.

If $\alpha = 1/2$, then (3.34) and (3.35) imply

$$\lim_{p\to\infty}\varphi_{2k}(p) = 0, \quad k = 2, \ldots,$$

and by (3.32), if the limit $\ell = \lim b_k c_k/k$ exists, then

$$\lim_{p\to\infty} \varphi_2(p) = \lim_{p\to\infty} \left(-\frac{b_p c_p}{2p+1}\right) = -\frac{\ell}{2},$$

so $tr(z) = -(\ell/2)z^2$. This completes the proof of Theorem 1.

6. Spectral Riemann Surfaces

In our analysis of the regularized trace, it was important to see that by inequality (4.38) from Lemma 10

$$a_k(n) = \frac{1}{k!}\left|E_n^{(k)}(0)\right| \leqslant (8M)^k n^{1-k(1-\alpha)},$$

so the series

$$\sum_{n=1}^{\infty} \left|E_n^{(k)}(0)\right| < \infty \quad \text{if } \alpha < 1 - 2/k. \tag{6.1}$$

Therefore, for every subset $\mathcal{M} \subset \mathbb{N}$, the partial sum

$$\mathcal{E}^{(k)}(\mathcal{M}) = \sum_{m\in\mathcal{M}} E_m^{(k)}(0) \tag{6.2}$$

is well defined.

On the other hand, (5.19) and the Cauchy inequality for the Taylor coefficients yield

$$\frac{1}{k!}\left|\widetilde{E}_n^{(k)}(0)\right| \leqslant \frac{2n^2}{(R_n/2)^k} = 2(16M)^k n^{2-k(1-\alpha)}. \tag{6.3}$$

So, if $\alpha < 1 - 2/k$, then

$$\lim_n \left|\widetilde{E}_n^{(k)}(0)\right| = \lim_n \left|\sum_{m\in\mathcal{M}, m\leqslant n} E_m^{(k)}(0)\right| = 0. \tag{6.4}$$

Therefore, the following statement is true.

Proposition 16. *If $\alpha < 1 - 2/k$, then we have for each equivalence class $\mathcal{M}$ of the SRS of the pair (L, B) that*

$$\mathcal{E}^{(k)}(\mathcal{M}) \equiv \sum_{m\in\mathcal{M}} E_m^{(k)}(0) = 0. \tag{6.5}$$

4. Finally, we show that some SRS are irreducible, which is the claim of Theorem 3.

Proof of Theorem 3. First, we consider the case where (1.3) holds with $\alpha = 1/2$. By Proposition 16, Theorem 3 will be proved if we show that there is no proper subset $\mathcal{M} \subset \mathbb{N}$ with property (6.5) for a fixed $k > 2/(1-\alpha)$. Indeed, then $\mathbb{N}$ will be the only one equivalence class, which implies that the SRS is irreducible.

If $\alpha = 1/2$, then $k = 6$ is the least even k for which $k > 2/(1-\alpha)$. By (4.49), we have

$$\frac{1}{k!} E_n^{(6)}(0) = a_6(1/2, n) = \psi(n) - \psi(n-1), \quad n = 2, 3, \dots, \tag{6.6}$$

where

$$\psi(n) = -\frac{1}{(2n-1)(2n+1)^5(2n+3)}. \tag{6.7}$$

On the other hand, from (4.24), with $\alpha = 1/2$, it follows that

$$a_6(1/2, 1) = \psi(1) = -\frac{1}{5 \cdot 3^5}. \tag{6.8}$$

In view of (6.6)–(6.8),

$$a_6(1/2, 1) < 0, \quad a_6(1/2, n) > 0 \quad \text{for } n \geqslant 2, \tag{6.9}$$

and

$$\sum_{n=2}^{\infty} a_6(1/2, n) = -a_6(1/2, 1).$$

Certainly, if

$$\sum_{n \in \mathcal{M}} a_6(n) = 0 \quad \text{then } \mathcal{M} = \mathbb{N}. \tag{6.10}$$

This proves Theorem 3 for $\alpha = 1/2$. □

If $\alpha \in [0, 1/2)$, then $4 > 2/(1-\alpha)$, so, in view of Proposition 16 and the above discussion, the SRS corresponding to $\alpha \in [0, 1/2)$ will be irreducible if all but one terms of the sequence

$$a_4(\alpha, n) = \frac{1}{4!} E_n^{(4)}(0)$$

have the same sign. Below, in Lemma 17, we show that this is true if $\alpha \in [0, 0.085]$ and $\alpha \in [(2-\sqrt{2})/4, 1/2]$, which completes the proof of Theorem 3.

5. For convenience, we set $\gamma = 2\alpha$ and

$$\tilde{a}_4(\gamma, n) = a_4(\gamma/2, n), \quad \tilde{\varphi}_4(\gamma, n) = \varphi_4(\gamma/2, n).$$

Lemma 17. *Under the above notations, we have*

$$\tilde{a}_4(\gamma, 1) = \tilde{\varphi}_4(\gamma, 1) = \frac{1}{27} - \frac{2^\gamma}{72} > 0, \quad \gamma \in [0, 1], \tag{6.11}$$

$$\tilde{a}_4(\gamma, 2) < 0, \quad \gamma \in [0, 1], \tag{6.12}$$

$$\tilde{a}_4(\gamma, n) > 0 \quad \textit{if } \gamma \in [0, 0.1717], \quad n \geqslant 3, \tag{6.13}$$

$$\tilde{a}_4(\gamma, n) < 0 \quad \textit{if } \gamma \in \left[\left(\sqrt{2} - 1\right)/\sqrt{2}, 1\right], \quad n \geqslant 3. \tag{6.14}$$

Proof. By (3.23) and (3.38), we have that (6.11) holds, and moreover,

$$\tilde{a}_4(\gamma, n) = \tilde{\varphi}_4(\gamma, n) - \tilde{\varphi}_4(\gamma, n-1), \tag{6.15}$$

where

$$\tilde{\varphi}_4(n) = \frac{n^{2\gamma}}{(2n+1)^3} - \frac{n^\gamma (n+1)^\gamma}{(2n+1)^2(4n+4)} - \frac{(n-1)^\gamma n^\gamma}{4n(2n+1)^2}. \tag{6.16}$$

In particular,

$$\tilde{a}_4(\gamma, 2) = \tilde{\varphi}_4(\gamma, 2) - \tilde{\varphi}_4(\gamma, 1) = \left(\frac{2^{2\gamma}}{5^3} - \frac{6^\gamma}{300} - \frac{2^\gamma}{200}\right) - \left(\frac{1}{27} - \frac{2^\gamma}{72}\right).$$

Graphing $\tilde{a}_4(\gamma, 2)$ one can easily see that (6.12) holds. In the same way, one can verify that the following inequalities hold:

$$\tilde{a}_4(\gamma, m) > 0 \quad \text{if } \gamma \in [0, 0.1717], \quad m = 3, 4, 5, 6, \tag{6.17}$$

and

$$\tilde{a}_4(\gamma, m) < 0 \quad \text{if } \gamma \in \left[\left(\sqrt{2} - 1\right)/\sqrt{2}, 1\right], \quad m = 3, 4, 5, 6. \tag{6.18}$$

In order to prove (6.13) and (6.14) for each $n > 6$, we study the sign of partial derivative $\partial\varphi_4/\partial n$. Set

$$b(\gamma, n) = n^{2-2\gamma}(2n+1)^2 \cdot \frac{\partial\tilde{\varphi}_4}{\partial n}(\gamma, n), \tag{6.19}$$

then

$$\begin{aligned} b(\gamma, n) = & -\frac{3}{2}\left(1 + \frac{1}{2n}\right)^{-2} + \gamma\left(1 + \frac{1}{2n}\right)^{-1} \\ & -\frac{\gamma}{4}\left(1 + \frac{1}{n}\right)^{\gamma-1} - \frac{c-1}{4}\left(1 + \frac{1}{n}\right)^{\gamma-2} + \frac{1}{2}\left(1 + \frac{1}{n}\right)^{\gamma-1}\left(1 + \frac{1}{2n}\right)^{-1} \\ & -\frac{\gamma}{4}\left(1 - \frac{1}{n}\right)^{\gamma-1} + \frac{1}{2}\left(1 - \frac{1}{n}\right)^{\gamma}\left(1 + \frac{1}{2n}\right)^{-1} - \frac{\gamma-1}{4}\left(1 - \frac{1}{n}\right)^{\gamma}. \end{aligned} \tag{6.20}$$

The power series expansion of $b(\gamma, n)$ about $n = \infty$ is

$$b(\gamma, n) = \sum_{k=2}^{\infty} b_k(\gamma)(1/n)^k, \tag{6.21}$$

where

$$b_2(\gamma) = \frac{5 - 22\gamma + 18\gamma^2 - 4\gamma^3}{8}, \tag{6.22}$$

$$b_3(\gamma) = \frac{-10 + 25\gamma - 14\gamma^2 + 2\gamma^3}{8}. \tag{6.23}$$

By (6.20), estimating from above $|b_k(\gamma)|$, we obtain

$$\begin{aligned} b_k(\gamma) \leqslant & \frac{3}{2} \cdot \frac{k+1}{2^k} + \frac{\gamma}{2^k} + \frac{\gamma}{4} + \frac{1-\gamma}{4}(k+1) \\ & + \frac{1}{2}\left(\frac{1}{2^k} + 2\gamma\right) + \frac{\gamma}{4} + \frac{1}{2}\left(\frac{1}{2^k} + \gamma\right) + \frac{1-\gamma}{4} \cdot \frac{\gamma}{k}, \end{aligned} \tag{6.24}$$

where each term comes from the expansion of the corresponding term in (6.20).

For example, consider

$$\left(1 - \frac{1}{n}\right)^{\gamma} \left(1 + \frac{1}{2n}\right)^{-1} = \left[1 + \sum_{i=1}^{\infty} \binom{\gamma}{i} \left(\frac{1}{n}\right)^i\right] \sum_{j=0}^{\infty} 2^{-j} \left(-\frac{1}{n}\right)^j. \tag{6.25}$$

Since $0 \leqslant \gamma \leqslant 1$, we have

$$\left| \binom{\gamma}{i} \right| = \frac{\gamma}{i} \cdot \frac{|\gamma - 1|}{1} \cdot \frac{|\gamma - 2|}{2} \cdots \frac{|\gamma - (i-1)|}{i-1} \leqslant \frac{\gamma}{i}. \tag{6.26}$$

Thus, the absolute value of the coefficient of $(1/n)^k$ in (6.25) does not exceed

$$\begin{aligned} & \frac{1}{2^k} + \frac{\gamma}{1} \cdot \frac{1}{2^{k-1}} + \frac{\gamma}{2} \cdot \frac{1}{2^{k-2}} + \frac{\gamma}{3} \cdot \frac{1}{2^{k-3}} + \cdots + \frac{\gamma}{k} \\ & \leqslant \frac{1}{2^k} + \frac{\gamma}{2}\left(\frac{1}{2^{k-2}} + \frac{1}{2^{k-2}} + \frac{1}{2^{k-3}} + \cdots + 1\right) = \frac{1}{2^k} + \gamma. \end{aligned}$$

Inequality (6.24) may be written as

$$b_k(\gamma) \leqslant \frac{3}{2} \cdot \frac{k+1}{2^k} + \frac{1+\gamma}{2^k} + 2\gamma + \frac{1-\gamma}{4}(k+1) + \frac{\gamma}{4k}. \tag{6.27}$$

Since

$$\sum_{k=4}^{\infty} x^k = \frac{x^4}{1-x}, \quad \sum_{k=4}^{\infty} (k+1) x^k = \left(\frac{x^5}{1-x}\right)' = \frac{5x^4 - 4x^5}{(1-x)^2},$$

we obtain, by (6.27),

$$\sum_{k=4}^{\infty} |b_k(\gamma)| n^{-k} \leqslant M(\gamma, n) \cdot \frac{1}{n^3}, \tag{6.28}$$

where

$$M(\gamma, n) = \frac{3(10n-4)}{16(2n-1)^2} + \frac{1+\gamma}{8(2n-1)} + \frac{33\gamma}{16(n-1)} + \frac{1-\gamma}{4} \cdot \frac{5n-4}{(n-1)^2}. \tag{6.29}$$

Thus, we have

$$n b_2(\gamma) + b_3(\gamma) - M(\gamma, n) \leqslant n^3 b(\gamma, n) \leqslant n b_2(\gamma) + b_3(\gamma) + M(\gamma, n). \tag{6.30}$$

On the other hand,

$$8b_2(\gamma) = (5-2\gamma)\left(\gamma - \left(1 - 1/\sqrt{2}\right)\right)\left(\gamma - \left(1 + 1/\sqrt{2}\right)\right),$$

and therefore,

$$b_2(\gamma) > 0 \ \text{ for } \gamma \in \left[0, 1 - 1/\sqrt{2}\right), \quad b_2(\gamma) < 0 \quad \text{for } \gamma \in \left(1 - 1/\sqrt{2}, 1\right]. \tag{6.31}$$

One can easily see, for each fixed $\gamma \in [0,1]$, that $M(\gamma, n)$ is a decreasing function of n. This fact leads, in view of (6.30) and (6.31), to the following inequalities:

$$0 < 6b_2(\gamma) + b_3(\gamma) - M(\gamma, 6) \leqslant n^3 b(\gamma, n), \quad \gamma \in [0, 0.19), \quad n \geqslant 6 \tag{6.32}$$

and

$$n^3 b(\gamma, n) \leqslant 6b_2(\gamma) + b_3(\gamma) + M(\gamma, 6) < 0, \quad \gamma \in \left(1 - 1/\sqrt{2}, 1\right], \quad n \geqslant 6. \tag{6.33}$$

(We checked the left inequality in (6.32) and the right inequality in (6.33) numerically by graphing the corresponding functions of γ.)

In view of (6.19) and (6.32), $\partial\tilde{\varphi}/\partial n(\gamma, n) > 0$ if $\gamma \in [0, 0.19]$ and $n \geqslant 6$, so $\tilde{\varphi}(\gamma, n)$ increases with n. Therefore, for each $\gamma \in [0, 0.19]$ and $n > 6$, we obtain by (6.17) that $\tilde{a}(\gamma, n) > \tilde{a}(\gamma, 6) > 0$, which proves (6.13).

In a similar way, (6.33) implies that $\tilde{\varphi}(\gamma, n)$ decreases with n if $\gamma \in [1 - 1/\sqrt{2}, 1]$ and $n \geqslant 6$. Thus, in view of (6.18), we obtain that $\tilde{a}(\gamma, n) < \tilde{a}(\gamma, 6) < 0$, for $\gamma \in [1 - 1/\sqrt{2}, 1]$ and $n \geqslant 6$, which proves (6.14). This completes the proof of Lemma 17. □

7. Conclusion; comments and questions

1. So far in our analysis we focused on the tri-diagonal matrices given by (2.1) and (2.2), or (1.3) with $\alpha < 1$, or even with $\alpha \leqslant 1/2$. The Whittaker–Hill matrices (1.6) satisfy (2.1) and (2.2) with $\alpha = 1$, $M = 4 + t$. Proposition 4 tells us that the eigenvalues $E_n(z)$, $n \in \mathbb{N}$, are analytic functions in the disk $\Delta = \{|z| < 1/(8M)\}$, and nothing more. But these matrices come from the differential operator

$$Ay = -y'' + q(x)y, \tag{7.1}$$

considered with

$$q(x) = a\cos 2x + b\cos 4x, \quad a = -4zt, \quad b = -2z^2. \tag{7.2}$$

Let $q(x)$ be a real analytic periodic function of period π. Of course, then q extends analytically in a neighborhood of $I = [0, \pi]$, say, in

$$G_\varepsilon = \{w = x + iy : -\varepsilon \leqslant x \leqslant \pi + \varepsilon, \ -\varepsilon \leqslant y \leqslant \varepsilon\}, \quad \exists \varepsilon > 0. \tag{7.3}$$

In other words, q is in the Banach space $A(G_\varepsilon)$ of all functions that are continuous in G_ε and analytic in its interior, with the norm

$$\|f\| = \max\{|f(w)| : w \in G_\varepsilon\}.$$

Consider the boundary conditions

$$Per^+ : y(0) = y(\pi), \quad y'(0) = y'(\pi),$$

$$Per^- : y(0) = -y(\pi), \quad y'(0) = -y'(\pi),$$

$$Dir : y(0) = y(\pi) = 0.$$

To be certain, let us talk only about the periodic boundary conditions Per^+, and let us consider the (invariant) subspace of even functions. Then, operator (7.1) has eigenvalue functions $E_n(z),\ E_n(0) = (2n)^2,\ n = 1, 2, \ldots$.

Volkmer [39] proved that if q is a real analytic function, then $E_n(z)$ is well defined as an analytic function in the disk

$$\Delta_n = \{z : |z| \leqslant R_n\}, \quad R_n = an^2, \ \ a > 0.$$

Careful analysis of the proof in [39] shows that a stronger quantitative statement holds.

Proposition 18. *If*

$$q \in A(G_\varepsilon), \quad \varepsilon > 0, \tag{7.4}$$

then the eigenvalues $E_n(q)$ of operator (7.1) *are well defined if q is real valued on* $[0, \pi]$ *and small by norm. Moreover, for each n, $E_n(q)$ can be extended as an analytic function of q in the ball*

$$B(R_n) = \{q \in A(G_\varepsilon) : \|q\| \leqslant R_n\},$$

with $R_n = an^2,\ a = a(\varepsilon) > 0$.

As soon as we have this proposition, we can consider the potentials (7.2) as elements of $A(G_\varepsilon)$, with, say, $\varepsilon = 1/4$. Then,

$$\|4zt\cos 2x + 2z^2\cos 4x\| \leqslant 4|zt|e^{2\cdot\frac{1}{4}} + 2|z|^2 e^{4\cdot\frac{1}{4}} \leqslant 7\left(|zt| + |z|^2\right), \tag{7.5}$$

and therefore, if

$$|tz| + |z|^2 \leqslant \frac{a}{7} n^2, \tag{7.6}$$

then

$$e_n(z) = E_n(q), \quad q = 4zt\cos 2x + 2z^2\cos 4x, \tag{7.7}$$

is an analytic function of z. Choose

$$R_n = n(1 + 4|t|/a)^{-1}, \tag{7.8}$$

then

$$z \in \Delta_n = \{z : |z| \leqslant R_n\} \ \ \Rightarrow \ \ (7.6), \tag{7.9}$$

and therefore, the function $e_n(z)$ is analytic in the disk Δ_n.

We explained the following statement (which is stronger than its analogue coming from Proposition 4).

Proposition 19. *Under conditions* (1.6), *the spectrum of operator* (1.1) *is discrete. The function* $e_n(z) \in (7.7)$ *is analytic in* $\Delta_n \in (7.9)$, *and*

$$e_n(0) = n^2, \quad |e_n(z) - n^2| \leqslant n \ \text{ if } z \in \Delta_n.$$

2. Of course, the claim of Proposition 19, with $R_n = an/(a+4|t|)$, is stronger than Proposition 4 with $R_n = 1/8$. This example, together with Remark 12, supports our belief that, for matrices $(L, B) \in (2.1)$, Proposition 4 can be significantly improved, so that to give analyticity of $E_n(z) \in$ (2.4), (2.5) in the disk $\Delta_n \in (2.3)$ with

$$R_n = bn^{2-\alpha}, \quad \exists b = b(\alpha) > 0. \tag{7.10}$$

If $\alpha = 0$ this is true, but again it comes from Volkmer's result [37,39] for the Mathieu differential operator which is unitary equivalent to matrices (1.3) and (1.4).

However, even in this case, no approach to the proof of this statement is known in the framework of matrix analysis.

3. Of course, if the Taylor expansion

$$E_n(z) = n^2 + \sum_{k=1}^{\infty} a_{2k}(n) z^{2k}$$

is known, then one may find the radius of convergence of $E_n(z)$ as

$$r_n = \left(\limsup_k |a_{2k}(n)|^{1/2k} \right)^{-1}.$$

Proposition 7 gives that

$$|a_{2k}(n)| \leqslant 8kn \cdot \left(\frac{8M}{n^{1-\alpha}} \right)^{2k}$$

for $(L, B) \in (2.1) + (2.2)$. However, if $B \in (1.3)$, i.e.,

$$b_k = c_k = k^{\alpha}, \quad 0 \leqslant \alpha < 2,$$

we believe that

$$|a_{2k}(n)| \leqslant n^{\gamma} \left(\frac{A}{n^{2-\alpha}} \right)^{2k}, \quad n = 1, 2, \ldots, \ \ \gamma > 0, \ \ A > 0. \tag{7.11}$$

Of course, (7.11) would imply (7.10).

4. Maybe, representation (3.6), (3.23) and (3.24) of Propositions 7 and 8 could be used in an attempt to get (7.11). But let us make a couple of elementary remarks to Propositions 7 and 8.

Remark 20. It was observed in (3.8), on the basis of the representation (3.6) and (3.23), (3.24), that $a_k(n) \equiv 0$ for odd k. *This follows also from the equality*

$$Sp(L + zB) = Sp(L - zB), \quad z \in \mathbb{C}, \tag{7.12}$$

because (7.12) *implies that all* $E_n(z)$ *are even functions.*

(In particular, this implies that in formulas like (4.58) and (4.59) the coefficients $P_k(z)$ should be even functions. In [4], however, formula (8) in Theorem 2.1 has $P_1(z) = (z^3 - 4z)/16$, so one can conclude that this is not correct even without knowing the correct formula.)

To get (7.12), consider the unitary operator U defined by

$$Ue_j = (-1)^j e_j, \quad 1 \leqslant j < \infty, \quad U^2 = 1. \tag{7.13}$$

Then, for each matrix $A = [A(i, j)]$ the operator $\widetilde{A} = U^{-1}AU = UAU$ has a matrix $\widetilde{A}(i, j) = (-1)^{i-j}A(i, j)$. In particular, $U^{-1}(L + zB)U = L - zB$, i.e., the operators $L + zB$ and $L - zB$ are similar, and therefore, (7.12) holds. Of course, this implies that $E_n(z)$ are even functions.

Remark 21. By Propositions 7 and 8, the integrals that appear in (3.8) and (3.24) vanish if $|j - n| > k$. But *they vanish even if* $|j - n| > k/2$.

After Remark 20, we can talk only about even k, say $k = 2m$. Let us focus on (3.24), i.e., on the integrals

$$I(n; j, k) = \int_{h_n} \lambda \langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j \rangle \, d\lambda, \tag{7.14}$$

where $h_n = \{\lambda \in \mathbb{C} : \lambda = n^2 + n + it, \ t \in \mathbb{R}\}$.

The integrand in (7.14) is a linear combination of rational functions like (3.26) with coefficients depending on B, where each rational function corresponds to a walk $(j_0, j_1, \ldots, j_k)$ from j to j on the integer grid $\mathbb{Z}$, with steps ± 1. Indeed, when the operator $R^0(BR^0)^k$ acts on e_j, then (since $R^0 e_\nu = (1/(\lambda - \nu^2))e_\nu$ while Be_ν is a linear combination of $e_{\nu-1}$ and $e_{\nu+1}$) we get a linear combination of 2^k vectors, each of them coming from some walk $(j_0, j_1, \ldots, j_k)$ as $e_{j_0} \to e_{j_1} \to \cdots \to e_{j_k}$. Since $\langle e_{j_k}, e_j \rangle \neq 0$ only for $j_k = j$, we consider further only walks from j to j.

Moreover, the argument used to prove the point (iii) in the proof of Proposition 8 shows that the rational function Q of (3.26) yields a non-zero integral over the line h_n only if it has poles both on the left and on the right of h_n, and its poles j_ν^2 come from the vertexes of the corresponding walk $(j_0, j_1, \ldots, j_k)$. In other words, if $j < n$ (respectively, $j > n$) then the corresponding walk $j_0 = j, j_1, \ldots, j_k = j$ should pass through $n + 1$ (respectively, n).

Take now any j such that $|j - n| > k/2$. If $j < n$ (respectively, $j > n$), then there is no k-step walk from j to j passing through $n + 1$ (respectively, n) because the steps are equal to ± 1. Thus, each of the integrals (7.14) vanishes if $|j - k| > k/2$.

5. We consider $\alpha \geqslant 0$ in (1.3) and elsewhere to have unbounded or non-compact operators B. Of course, Theorems 1 and 2 remain valid for $\alpha < 0$ as well. But then a simpler proof can be given because for $\alpha < 0$ the restriction $\alpha < 1 - 2/k$ holds with $k = 2$. In particular, by (6.5), i.e., by Proposition 16, we have

$$\mathcal{E}^{(2)}(\mathcal{M}) = \sum_{m \in \mathcal{M}} E_m^{(2)}(0) = 0$$

for any equivalence class of the SRS of the pair $(L, B) \in (2.1) + (2.2), \ \alpha < 0$.

Of course, it is easier to study the sign of $a_2(\alpha, n)$ than the sign of $a_4(\alpha, n)$ (compare to Lemma 9). By (3.32), we have

$$a_2(1) = -\frac{b_1 c_1}{3}, \quad a_2(n) = \frac{b_{n-1}c_{n-1}}{2n-1} - \frac{b_n c_n}{2n+1}.$$

If (b_n) and (c_n) are decreasing sequences of positive numbers, then

$$a_2(1) < 0, \quad a_2(n) > 0, \quad n \geqslant 2,$$

and we can use the same argument as before (see the proof of Theorem 3) to conclude that the corresponding SRS is irreducible. So, we proved the following analog of Theorem 3.

Proposition 22. *Suppose that* (2.1) *and* (2.2) *hold with monotone decreasing sequences* $b = (b_n)$ *and* $c = (c_n)$, *and with* $\alpha < 0$. *Then, the corresponding SRS is irreducible.*

We have to admit that with all variety of pairs (L, B) for which we have proved the SRS's irreducibility, we know no nontrivial (i.e., beside the case where some entries b_k or c_k vanish, or diagonal entries are multiple) example of a pair (L, B) with a reducible SRS.

6. From $\alpha < 0$ we can go to another direction, i.e., consider $\alpha \in (1/2, 1)$. The estimate (7.11) is our conjecture, but even now we can claim the following amendment to Theorem 1.

Proposition 23. *Under assumptions* (2.1) *and* (2.2), *if* $0 \leqslant \alpha < 9/10$, *then the regularized trace*

$$tr_1(z) = \sum_{n=1}^{\infty} \left(E_n(z) - n^2 - \frac{1}{2} E_n''(0) z^2 \right) \tag{7.15}$$

is well defined as an entire function of z, *and*

$$tr_1(z) \equiv 0. \tag{7.16}$$

The proof is based on (4.41)–(4.44) and the estimates given by Lemma 9. It goes along the same lines as Definition of regularized trace in Section 5.4 and the proof of Theorem 1; see (5.20)–(5.22). We omit the details.

Of course, one can introduce the higher order regularized traces

$$tr_p(z) = \sum_{n=1}^{\infty} \left(E_n(z) - n^2 - \sum_{j=1}^{p} \frac{E_n^{(2j)}(0)}{(2j)!} z^{2j} \right)$$

and study for which α this expression is well defined as an entire function.

It is important to mention that many interesting examples of evaluation of a regularized trace can be found in the recent papers [7–9,16,21–24], although there the operators L and B are usually self-adjoint and z is real. Let us notice that, in our Theorem 1, the first line of (1.7), $\alpha < 1/2$, can be interpreted as an example to Theorem 1 in [22]. Then, the second line of (1.7) shows that the restrictions on δ and ω in [22] could not be weakened.

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